# Eigenvalues of the Anti-periodic Calogero - Sutherland Model

Arindam Chakraborty, Subhankar Ray *
Dept. of Physics, Jadavpur University, Kolkata 700032, India

5 October 2004

**Abstract**

The $U(1)$ Calogero Sutherland Model (CSM) with anti-periodic boundary condition is studied. The Hamiltonian is reduced to a convenient form by similarity transformation. The matrix representation of the Hamiltonian acting on a partially ordered state space is obtained in an upper triangular form. Consequently the diagonal elements become the energy eigenvalues.

## 1 Introduction

In recent years many exactly solvable models have been proposed for quantum many body systems in one dimension. The basic advantage of dealing with such one dimensional problem is that due to highly restrictive spatial degrees of freedom, there exist several algebraic techniques that are applicable. The well studied Bethe Ansatz technique is used to solve models with nearest neighbour exchange interaction [1]. But for long range interaction a few of the recently proposed models prove to have direct correspondence with the physical reality.

Recently there have been much interests in the class of models with the interaction that falls off as a inverse square of the distance between a pair of particles or spins. The study of exact features of long-range interaction in this type of model dates back to Moser [2], Calogero [3], Sutherland [4, 5]. The Hamiltonian of the Calogero Sutherland Model (CSM) in units of $\hbar^2/2m$ is given by,

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x^2} + \sum_{j<k} \frac{2\alpha(\alpha-1)}{d^2(x_j - x_k)} \tag{1}$$

where $\alpha$ is the interaction parameter and $d(x)$ is the length between sites,

$$d(x) = \frac{L}{\pi} \sin\left(\frac{\pi x}{L}\right)$$

$L$ is the total length of the one dimensional chain.

One of the most important practical features of the model is that the inverse square potential can be regarded as a pure statistical interaction and the model maps to an ideal gas of particle obeying the fractional statistics [6, 7, 8].

*Email: subho@juphys.ernet.in

The eigenstate of $U(1)$ CSM can be written in terms of Jack symmetric polynomial [8, 9] whose algebraic properties provide a powerful and direct method of calculating the most general correlation function. The exact calculation of the correlation function further provide conclusive evidences of the inherent fractional exclusion and exchange statistics embodied in CSM [10, 9]. The CSM is directly related to several interesting physical and mathematical problems, e.g., Selberg integral [10, 9], $W^{\infty}$ algebra [12], edge states of quantum Hall droplet [8, 7], random matrix theory [20, 21], Jack symmetric polynomials [22].

In our present paper we have calculated the energy eigenvalues of the CSM Hamiltonian with anti-periodic boundary condition. Such boundary condition is relevant where a magnetic field pierces perpendicular to the plane of one dimensional ring (Topological representation of one dimensional linear chain of sites). When a particle goes around the entire system $n$ times it picks up a net phase $\exp(in\phi)$. Consequently the interaction term takes the following form [23],

$$\sum_{n=-\infty}^{+\infty} \frac{\exp\left(i\phi n\right)}{(x+nL)^2} \tag{2}$$

where $L =$ length of the chain. Let $\phi = 2\pi p/q$ and $n = jq + k$, where $p$ and $q$ are mutual primes and $j$ and $k$ are integer such that

$$-\infty < j < +\infty, \qquad\qquad 0 \leq k \leq q-1.$$

Therefore, the sum in the above expression, Eq.(2) can be rewritten as,

$$\sum_{k=0}^{q-1} \sum_{j=-\infty}^{+\infty} \frac{1}{\left[(x+kL)+(qL)j\right]^2} = \sum_{k=1}^{q-1} \frac{\exp\left(i2\pi px/q\right)}{\left[(qL/\pi)\sin\frac{\pi(x+kL)}{qL}\right]^2} \tag{3}$$

The effective length of the chain is thus $(qL)$. The above interaction term represents in general the so called twisted boundary condition [23]. When $p/q = 1/2$ the corresponding particle interaction term becomes,

$$\frac{\cos\left(\pi x/L\right)}{\frac{L^2}{\pi^2}\sin^2\left(\pi x/L\right)} \tag{4}$$

Eq.(4) gives the interaction term with anti-periodic boundary condition. Under such boundary condition the final expression of the model Hamiltonian becomes,

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial {x_j}^2} + 2\alpha(\alpha-1)\frac{\pi^2}{L^2}\sum_{j<k} \frac{\cos\left[(\pi/L)(x_j-x_k)\right]}{\sin^2\left[(\pi/L)(x_j-x_k)\right]} \tag{5}$$

The diagonalization of such Hamiltonian is usually done after reducing it to a convenient form by means of successive similarity transformations. Here the final form of the Hamiltonian is diagonalized with the help of the symmetric polynomial type eigenfunctions, known as the Jack symmetric polynomials. The matrix representation of the Hamiltonian becomes triangular on a partially ordered state space. The action of the Hamiltonian on such a state space gives rise to several *mother* and *daughter* states. The mother and daughter states are connected by a particular rule. And a topological representation of the excitation spectrum is obtained for a specific example.

## 2 Simplification of the model Hamiltonian

### 2.1 Similarity transformation

Let us put ($\lambda = -\alpha$) in Eq.(5). The the expression for the Hamiltonian becomes,

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial {x_j}^2} + 2\lambda(\lambda+1)\frac{\pi^2}{L^2}\sum_{j<k}\frac{\cos\left[(\pi/L)(x_j-x_k)\right]}{\sin^2\left[(\pi/L)(x_j-x_k)\right]} \tag{6}$$

Eq.(6) can be rewritten in the following form, apart from some constant terms.

$$\begin{aligned} H = &-\sum_{j=1}^{N} \frac{\partial^2}{\partial {x_j}^2} + 4\lambda\frac{\pi^2}{L^2}\sum_{j<k}\frac{\cos\left[\frac{\pi}{L}x_{jk}\right]}{\sin^2\left[\frac{\pi}{L}x_{jk}\right]} + 2\lambda(1-\lambda)\frac{\pi^2}{L^2}\sum_{j<k}\frac{\cos[\frac{\pi}{L}x_{jk}]+1}{\sin^2[\frac{\pi}{L}x_{jk}]} \\ &-2\lambda(1-\lambda)\frac{\pi^2}{L^2}\sum_{j<k}\left(\frac{1}{\sin^2\left[\frac{\pi}{L}x_{jk}\right]}-1\right) + 4\lambda(\lambda-1)\frac{\pi^2}{L^2}\sum_{j<k}\frac{\cos[\frac{\pi}{L}x_{jk}]}{\sin^2\left[\frac{\pi}{L}x_{jk}\right]} \end{aligned} \tag{7}$$

where $x_{jk} = x_j - x_k$.

Putting $\omega_j = \exp(i\pi x_j/L)$, where $i = \sqrt{-1}$, Eq.(7) becomes,

$$\begin{aligned} H = \frac{\pi^2}{L^2}\Bigg[&\sum_{j=1}^{N}(\omega_j\frac{\partial}{\partial\omega_j})^2 + 4\lambda\sum_{j<k}\left(\omega_j\frac{\partial}{\partial\omega_j}-\omega_k\frac{\partial}{\partial\omega_k}\right)\frac{\omega_j\omega_k}{\omega_j^2-\omega_k^2} \\ &-4\lambda(1-\lambda)\sum_{j<k}\frac{\omega_j\omega_k}{(\omega_j-\omega_k)^2} + 2\lambda(1-\lambda)\sum_{j<k}\left(\frac{\omega_j^2+\omega_k^2}{\omega_j^2-\omega_k^2}\right)^2 \\ &-8\lambda(\lambda-1)\sum_{j<k}\frac{\omega_j^2+\omega_k^2}{(\omega_j^2-\omega_k^2)^2}\;\omega_j\omega_k\Bigg] \end{aligned} \tag{8}$$

Let's take a similarity transformation with the following ansatz,

$$\psi = \prod_{j<k}\left(\frac{\omega_j}{\omega_k}-\frac{\omega_k}{\omega_j}\right)^{\lambda}\phi \tag{9}$$

Therefore, the Hamiltonian in units of $\pi^2/L^2$ becomes,

$$\begin{aligned} \hat{H} = &\sum_{j=1}^{N}\left[\omega_j\frac{\partial}{\partial\omega_j}\right]^2 + 2\lambda\sum_{j<k}\left(\frac{\omega_j+\omega_k}{\omega_j-\omega_k}\right)\left[\omega_i\frac{\partial}{\partial\omega_j}-\omega_k\frac{\partial}{\partial\omega_k}\right] \\ &+4\lambda(\lambda-1)\sum_{j<k}\frac{\omega_j\omega_k}{(\omega_j-\omega_k)^2} \end{aligned} \tag{10}$$

Again using the similarity

$$\phi = \prod_{j<k}\left(\frac{\omega_j}{\omega_k}+\frac{\omega_k}{\omega_j}-2\right)^{\beta/2}\phi_0 \tag{11}$$

where $\beta = \beta(\lambda)$. The above Hamiltonian takes the form

$$\tilde{H} = \sum_{j=1}^{N}\left[\omega_j\frac{\partial}{\partial\omega_j}\right]^2 + (2\lambda+\frac{3}{2}\beta(\lambda))\sum_{j<k}\left(\frac{\omega_j+\omega_k}{\omega_j-\omega_k}\right)\left[\omega_i\frac{\partial}{\partial\omega_j}-\omega_k\frac{\partial}{\partial\omega_k}\right] \tag{12}$$

where

$$\beta(\lambda) = \frac{1}{2}\left(-1\pm\sqrt{1+8\lambda-8\lambda^2}\right) \tag{13}$$

Introducing $A(\lambda) = 2\lambda + (3/2)\beta(\lambda)$, we get the following expression for the model-Hamiltonian,

$$\tilde{H} = \sum_{j=1}^{N} \left[\omega_j \frac{\partial}{\partial \omega_j}\right]^2 + A(\lambda) \sum_{j<k} \left(\frac{\omega_j + \omega_k}{\omega_j - \omega_k}\right) \left[\omega_i \frac{\partial}{\partial \omega_j} - \omega_k \frac{\partial}{\partial \omega_k}\right] \tag{14}$$

Hence $\tilde{H}$ has two additive parts, one representing free particle Hamiltonian, and a second part containing the reduced interaction term.

$$\tilde{H} = H_0 + A(\lambda) H_1 \tag{15}$$

### 2.2 Action of the Hamiltonian on ordered states

The eigenstates of the present Hamiltonian, Eq.(15), can be given by the following Bosonic state,

$$|n_1 .... n_N\rangle = \sum_P \prod_{j=1}^{N} \omega_j^{n_{P_j}} \tag{16}$$

The set $\{n_i | i = 1...N\}$ can be considered as a Bosonic quantum number with no restriction on their values. Without loss of generality we can introduce an ordering $(n_1 \geq n_2 \geq n_3 \geq .... \geq n_N)$. The action of $H_0$ and $H_1$ on such an ordered state gives,

$$H_0 |n_1 .... n_N\rangle = \left(\sum_{j=1}^{N} n_j^2\right) |n_1 .... n_N\rangle \tag{17}$$

and

$$H_1 |n_1 .... n_N\rangle = \sum_{j<k} (n_j - n_k) \left(|n_1 .... n_N\rangle + 2 \sum_{p=1}^{n_j - n_k - 1} |, \ldots, n_j - p, \ldots, n_k + p, \ldots\rangle\right) \tag{18}$$

## 3 The Eigenvalues of the Hamiltonian

### 3.1 Level of states

We define $|n_1 ... n_N\rangle$ as the mother state and the states generated by squeezing a pair of quantum number by one unit like $| \ldots, n_j - 1, \ldots, n_k + 1, \ldots\rangle$ as the daughter state. Squeezing of mother state into daughter states is permitted when such squeezing retains ordering of $n_j$s, i.e, when $(n_j - n_k \geq 2)$.

The family of states can be organized into levels such that the members of a given level are mutually unrelated in a sense that they are unreachable from each other. The daughter of a member from any given level belongs to a lower level in the family. Let, $|u\rangle_1$ implies the highest level mother state. The state index $u$ represents the total number of levels in the family. $|u\rangle_\mu$ represents the family of daughter state with $1 \leq u' \leq u$ and $\mu =$ is an index for the state in the $u$th level.

## 3.2 Generation of Mother and Daughter states

Let us take $|6,4,3,1\rangle$ as the mother state. It is obvious that by squeezing a relevant pair of quantum numbers we obtain a sequence of mother and daughter states. Finally we reach an irreducible daughter state that cannot serve as a mother in order to produce daughter state(s). This irreducible state is called the ground state, and is commonly denoted by $|1\rangle_1$. Table.I shows the possible mother and daughter states starting with $|6,4,3,1\rangle$ as the highest level mother state.

Table.I: Mother and daughter state association.

| Mother State | Daughter State |
|---|---|
| $\|6,4,3,1\rangle = \|6\rangle_1$ | $\|5,5,3,1\rangle = \|5\rangle_1$<br>$\|6,4,2,2\rangle = \|5\rangle_2$<br>$\|5,4,4,1\rangle = \|4\rangle_1$<br>$\|6,3,3,2\rangle = \|4\rangle_2$<br>$\|5,4,3,2\rangle = \|3\rangle_1$ |
| $\|5,5,3,1\rangle = \|5\rangle_1$ | $\|5,4,4,1\rangle = \|4\rangle_1$<br>$\|5,5,2,2\rangle = \|4\rangle_3$<br>$\|5,4,3,2\rangle = \|3\rangle_1$ |
| $\|6,4,2,2\rangle = \|5\rangle_2$ | $\|6,3,3,2\rangle = \|4\rangle_2$<br>$\|5,5,2,2\rangle = \|4\rangle_3$<br>$\|5,4,3,2\rangle = \|3\rangle_1$ |
| $\|5,4,4,1\rangle = \|4\rangle_1$ | $\|5,4,3,2\rangle = \|3\rangle_1$<br>$\|4,4,4,2\rangle = \|2\rangle_1$ |

| Mother State | Daughter State |
|---|---|
| $\|6,3,3,2\rangle = \|4\rangle_2$ | $\|5,4,3,2\rangle = \|3\rangle_1$<br>$\|5,3,3,3\rangle = \|2\rangle_2$ |
| $\|5,5,2,2\rangle = \|4\rangle_3$ | $\|5,4,3,2\rangle = \|3\rangle_1$ |
| $\|5,4,3,2\rangle = \|3\rangle_1$ | $\|4,4,4,2\rangle = \|2\rangle_1$<br>$\|5,3,3,3\rangle = \|2\rangle_2$<br>$\|4,4,3,3\rangle = \|1\rangle_1$<br>(irreducible daughter state) |
| $\|4,4,4,2\rangle = \|2\rangle_1$ | $\|4,4,3,3\rangle = \|1\rangle_1$<br>(irreducible daughter state) |
| $\|5,3,3,3\rangle = \|2\rangle_2$ | $\|4,4,3,3\rangle = \|1\rangle_1$<br>(irreducible daughter state) |

Moreover as the states in the same level are not connected, the Hamiltonian is diagonal in that subspace. The following topological representation shows the connection of mother and daughter state in the excitation spectrum. Let us define $\nu(\eta)$ as the multiplicity of a number in a given state ket taken as a mother state. The weight of an arrow $W$ is given by the following equation,

$$W = \nu(\eta_i)\nu(\eta_j) \tag{19}$$

The following table gives the respective weights of the possible transitions from the mother to daughter states.

Table.II: Transitions and weights for Calogero-Sutherland model with anti-periodic boundary condition

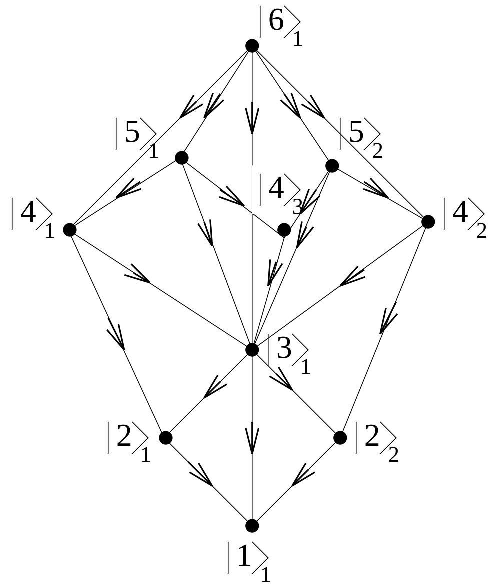


Figure 1: The topological representation of the excitation spectrum

| Transitions | Weights (W) |
|---|---|
| $\|6\rangle_1 = \|5\rangle_1$ | $1 \cdot 1 = 1$ |
| $\|6\rangle_1 = \|5\rangle_2$ | $1 \cdot 1 = 1$ |
| $\|6\rangle_1 = \|4\rangle_1$ | $1 \cdot 1 = 1$ |
| $\|6\rangle_1 = \|4\rangle_2$ | $1 \cdot 1 = 1$ |
| $\|6\rangle_1 = \|3\rangle_1$ | $1 \cdot 1 = 1$ |
| $\|5\rangle_1 = \|4\rangle_1$ | $2 \cdot 1 = 2$ |
| $\|5\rangle_1 = \|4\rangle_3$ | $1 \cdot 1 = 1$ |
| $\|5\rangle_1 = \|3\rangle_1$ | $2 \cdot 1 = 2$ |
| $\|5\rangle_2 = \|4\rangle_2$ | $1 \cdot 2 = 2$ |
| $\|5\rangle_2 = \|4\rangle_3$ | $1 \cdot 1 = 1$ |
| $\|5\rangle_2 = \|3\rangle_1$ | $1 \cdot 2 = 2$ |

| Transitions | Weights (W) |
|---|---|
| $\|4\rangle_1 = \|3\rangle_1$ | $2 \cdot 1 = 2$ |
| $\|4\rangle_1 = \|2\rangle_1$ | $1 \cdot 1 = 1$ |
| $\|4\rangle_2 = \|3\rangle_1$ | $1 \cdot 2 = 2$ |
| $\|4\rangle_2 = \|2\rangle_2$ | $1 \cdot 1 = 1$ |
| $\|4\rangle_3 = \|3\rangle_1$ | $2 \cdot 2 = 4$ |
| $\|3\rangle_1 = \|2\rangle_1$ | $1 \cdot 1 = 1$ |
| $\|3\rangle_1 = \|2\rangle_2$ | $1 \cdot 1 = 1$ |
| $\|3\rangle_1 = \|1\rangle_1$ | $1 \cdot 1 = 1$ |
| $\|2\rangle_1 = \|1\rangle_1$ | $3 \cdot 1 = 3$ |
| $\|2\rangle_2 = \|1\rangle_1$ | $1 \cdot 3 = 3$ |

### 3.3 Sub-family of states

Let us introduce the so called sub-family of states which consists of the highest level mother state and all her reachable daughter states. The total number of sub-family is called the dimension of the family. Since the action of the Hamiltonian on a given state space generates states belonging to lower levels, the matrix representation of the Hamiltonian in a partially ordered state space is always triangular. So the diagonal terms of the Hamiltonian matrix are the energy eigenvalues.

The energy of an eigen-state spanned by a family with the highest level mother state $|n_1...n_N\rangle$ is given by

$$E^0(n_1, \ldots, n_N) = \sum_{j=1}^{N} n_j^2 + A(\lambda) \sum_{j<k} (n_j - n_k) \tag{20}$$

The off diagonal elements are found to be

$$
{}_{\mu'}\langle u'|\tilde{H}|u\rangle_{\mu} = \begin{cases} \sum_P \left(\prod_{i\in P} W_i\right) E^1(n_1,\ldots,n_N) & \forall \mu > \mu' \\ \\ 0 & \forall \mu < \mu' \end{cases} \tag{21}
$$

with

$$
E^1(n_1,\ldots,n_N) = 2A(\lambda)\sum_{j<k}(n_j - n_k) \tag{22}
$$

The sum in Eq.(21) is over all possible paths $P$ from $|u\rangle_\mu$ to $|u'\rangle_{\mu'}$ and the product is over all weights $W_i$ of the intermediates arrows belonging to $P$. Then the general matrix element of the Hamiltonian is given by,

$$
{}_{\mu'}\langle u'|\tilde{H}|u\rangle_{\mu} = \varepsilon^{u,\mu}_{u',\mu'} \tag{23}
$$

# 4 Conclusion

In this article we obtain the energy eigen-value and eigen-states of the CSM with anti-periodic boundary condition (APBC). The model Hamiltonian, Eq.(5), is reduced to a convenient from by means of similarity transformation. Finally we obtain an upper triangular representation of the Hamiltonian in terms of a partially ordered basis. The eigenfunctions are chosen to be symmetric polynomials, known as the Jack polynomial.

# Acknowledgement

AC wishes to acknowledge the Council of Scientific and Industrial Research, India (CSIR) for fellowship support.

# References

[1] H.A. Bethe, Z. Phys. 71 (1931) 205.

[2] J. Moser, Adv. Math. 16 (1975) 197.

[3] F. Calogero, J. Math. Phys. 3 (1962) 2191, 10 (1969) 2197.

[4] B. Sutherland, J. Math. Phys. 12 (1971) 246, 12 (1971) 251.

[5] B. Sutherland, Phys. Rev. A 4 (1971) 2019, 5 (1972) 1372.

[6] Z. N. C. Ha, Phys. Rev. Lett. 73 (1994) 1574.

[7] Z. N. C. Ha, Nuc. Phys. B 435 [FS] (1995) 604.

[8] F. D. M. Halden, in the Proceeding of the 16th Taniguchi Symposium, Kashiko-Jima, Japan, Oct 26-29 (1993).

[9] P. J. Forrester, Nuc. Phys. B 416 (1994) 377.

[10] P. J. Forrester, Phys. Lett A 179 (1993) 127.

[11] P. J. Forrester, Nuc. Phys. B 388 (1992) 671.

[12] K. Hikami, M. Wadati, J. Phys. Soc. Jpn. 62 (1993) 1203.

[13] F. D. M. Halden, Bull. Am. Phys. Soc. 37 (1992) 164.

[14] F. D. M. Halden, Phys. Rev. Lett. 67 (1991) 937.

[15] S. Mitra, A. H. MacDonald, Bull. Am. Phys. Soc. 37 (1992) 377.

[16] P. J. Forrester, B. Jancovici, J. Phys. (Paris) 45 (1984) L583.

[17] A. P. Polychronakos, Nuc. Phys. B 324 (1989) 597.

[18] N. Kawakami, Phys. Rev. Lett. 71 (1993) 275.

[19] A. D. deVeigy, S. Ouvry, Phys. Rev. Lett. 72 (1994) 121.

[20] F. J. Dyson, J. Math. Phys. 3 (1962) 140, 157, 166.

[21] M. L. Mehta, 'Random Matrices', Academic Press, New York (1967).

[22] H. Jack, Proc. Roy. Soc. (Edinburgh) A 69 (1969) 1.

[23] Z. N. C. Ha, 'Quantum Many-Body Systems in One Dimension', World Sc., Singapore (1996).

[24] I. G. Macdonald, Lecture notes in Math. 1271 (1987) 189.

[25] I. G. Macdonald, *Symmetric functions and Hall Polynomials*, Clarendon Press (Oxford) 2nd Ed. (1995).

[26] R. P. Stanley, Adv. Math. 77 (1989) 76.